General Form of the Stiffness Matrix of a Tapered Beam-column

Yasha H. Zeinali, S. Milad Jamali, and Saman Musician

***Abstract*—**Shear deformation and distributed or concentrated axial force influence the structural behavior of all kinds of beams specially tapered ones. A tapered Timoshenko-Euler beam element (general form of beams) under a concentrated and constant distributed axial force with box-shaped cross-section is considered and governing equilibrium differential equation established. By utilizing the Chebyshev Polynomial approach technique, the governing differential equation is fully solved and stiffness matrix established.

***Keywords*—** Buckling Load, Chebyshev Polynomial, Stiffness Matrix, Tapered beams, Timoshenko-Euler beam-columns.

I. Introduction

STRUCTURAL members may have non-prismatic sections. Utilizing the non-prismatic members causes the stresses distribute more smoothly and the consumption of material reduces [1]. These facts lead designers to use widely tapered members.

During the past decades, several researchers have examined the structural behavior of this type of elements. Some of them focused on flexural or torsional buckling of non-prismatic members [4], [6], [8], [9], [10], [11], and some other focused on free vibration analysis of tapered members [5], [7], [12].

One of the analysis methods of tapered members is dividing it into a number of uniform elements which is called "step representation" [13]. This technique is lowly efficient [1]. An efficient method is solving the governing differential equation by using a numerical method.

According to modeling assumptions, four types of elements for tapered members exist which are illustrated in Table I.

TABLE I
TYPES OF BEAMS-ACCORDING TO CONSIDERED EFFECTS

Beam name	Axial Force	Shear Deformation
Bernoulli		
Bernoulli-Euler	×	
Timoshenko		×
Timoshenko-Euler	×	×

According to Table. I, Timoshenko–Euler beam element (the last one) includes effects of axial force and shear deformation. The Timoshenko–Euler element is the most general one.

Shear deformation and distributed or concentrated axial force have been verified to influence significantly the structural behavior of all kinds of beams specially tapered ones [14]. These influences reduce the stiffness of members.

Several researches have been done to establish the stiffness matrix and governing differential equation of equilibrium of a tapered beam element. In this studies a number of numerical methods have been proposed like: VIM (Variational Iteration Method) [17]-[19], General Polynomial Shape Function [4], Direct Integral approach [20], [21] and Bessel Function approach [1]. However, the governing differential equation of equilibrium in these publications did not simultaneously include effects of axial force and shear deformations.

Shear Deformations have been included in many researches but this studies were on free vibration analysis of Timoshenko tapered beams. The effects of axial force on elemental stiffness in these studies were excluded.

In 2002, Li and Li [1], established the equilibrium differential equation of a tapered Timoshenko-Euler beam by considering simultaneously effects of constant axial force and shear deformation. Li and Li employed power series method to obtain the expression of the tapered Timoshenko-Euler beam stiffness equation. In their equations, they assumed that the beam is just under a concentrated axial force and has an I-shaped section. They did not consider distributed axial force. So, to the authors' knowledge there is no publication at present obtaining the elemental stiffness matrix for tapered Timoshenko-Euler beam with box-shaped cross-section considering effects of concentrated and distributed axial force and shear deformation simultaneously.

This paper drives the governing differential equation of equilibrium of the Timoshenko-Euler beam axial force by considering the effects of concentrated or constant distributed axial force and shear deformation simultaneously.

II. Governing Differential Equation of Equilibrium

In this paper, it assumed that cross section of tapered beam is box-shape and symmetric. The applied forces as well as the corresponding deformation of tapered element is modeled as shown in Fig. 1 assuming the axis of the member remains straight.

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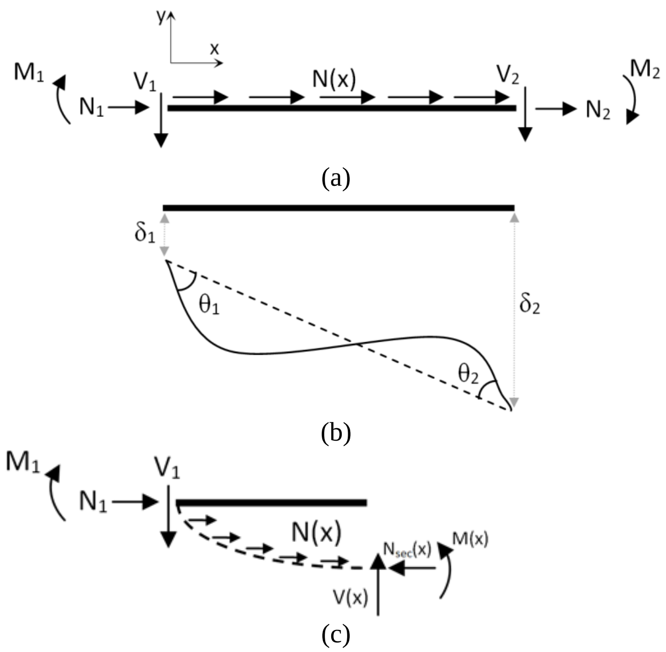

Fig. 1 (a) Element forces (b) Element deformation (c) Element free-body diagram

Fig. 1-a and 1-b shows the positive directions of forces and deformations. Element deflection (y) (shown in Fig. 1-b) consists of two portions. One is caused by bending deformation (y_M) and the other one induced by shear deformation (y_V). Thus we have

$$y = y_M + y_V \tag{1}$$

And a similar relation can be written for curvatures

$$y" = y"_M + y"_V \tag{2}$$

The curvature of the element caused by bending is

$$y"_M = \frac{-M}{E.I(x)} \tag{3}$$

In (3) $I(x)$ is inertia moment of the cross-section at the location of distance x from left end of the element. E is the elastic modules and M is The cross-section moment. M can be obtained by (4)

$$M(x) = M_1 - V_1.x + \int_0^x y(x)N(x)dx + N_1.y(x) \tag{4}$$

In (4), V_1 and M_1 are the left end shear force and moments and $N(x)$ and N_1 are distributed and concentrated axial force respectively. In box-shaped sections most of the shear stresses $\tau(x)$ are acting on webs. According to this fact, the shear stress approximately is equal to

$$\tau(x) = \frac{V(x)}{A_{Shear}(x)} = \frac{V(x)}{A_W(x)} = \frac{1}{A_W(x)} \cdot [\frac{d}{dx}M(x)] \tag{5}$$

In (5), $V(x)$ is the section shear force and $A_W(x)$ is web area. According to basic relations we have

$$\gamma(x) = \frac{\tau(x)}{G} = \frac{1}{G.A_W(x)} \cdot [\frac{d}{dx}M(x)] \tag{6}$$

In (6), G is the shear Modules. In box-section thin-walled members it can be seen that the slope of the member induced by shear deformation is equal to total section shear strain $\gamma(x)$. So the slope of the member caused by shear deformation (y'_V) is

$$y'_V(x) = \frac{1}{G.A_W(x)} \cdot [\frac{d}{dx}M(x)] \tag{7}$$

Substituting (4) into (7), we have

$$y'_V(x) = \frac{-V_1 + y(x)N(x) + N_1\, y'(x)}{G.A_w(x)} \tag{8}$$

Differentiating (8) by x yields

$$y"_V(x) = \frac{1}{G\,(A_w(x))^2}\left[V_1\left(\frac{d}{dx}A_w(x)\right) - y(x)N(x)\frac{d}{dx}A_w(x) - \left(\frac{d}{dx}A_w(x)\right)N_1\left(\frac{d}{dx}y(x)\right) + A_w(x)\left(\frac{d}{dx}y(x)\right)N(x) + A_w(x)y(x)\left(\frac{d}{dx}N(x)\right) + A_w(x)N_1\left(\frac{d^2}{dx^2}y(x)\right)\right] \tag{9}$$

And by substituting (4) into (3) we obtain

$$y"_M = \frac{-M_1 + V_1.x - \int_0^x y(x)N(x)dx - N_1.y(x)}{E.I(x)} \tag{10}$$

Substituting (9) and (10) into (2) and simplifying yields

$$\frac{d^2}{dx^2}y(x)\left(\frac{N_1}{G\,A_w(x)} - 1\right) + \frac{d}{dx}y(x)\Big(-\frac{\left(\frac{d}{dx}A_w(x)\right)N_1}{G\,(A_w(x))^2} + \frac{N(x)}{G\,A_w(x)}\Big) + y(x)\Big(-\frac{\left(\frac{d}{dx}A_w(x)\right)N(x)}{G\,(A_w(x))^2} + \frac{\frac{d}{dx}N(x)}{G\,A_w(x)} - \frac{N_1}{E\,I(x)}\Big) - \frac{\int_0^x y(x)N(x)dx}{E\,I(x)} = -\frac{V_1}{G\,(A_w(x))^2}\left(\frac{d}{dx}A_w(x)\right) + \frac{M_1}{E\,I(x)} - \frac{V_1\,x}{E\,I(x)} \tag{11}$$

(11) is the governing differential equation of equilibrium. In this paper the Chebyshev Polynomial approach have been utilized to solve this differential equation. It is better to solve the equation in the [0,1] domain [3]. So we need to convert the differential equation to non-dimensional form. This Procedure can be done by letting the $\xi = \frac{x}{L}$ and rewriting (11) by this assumption. By defining some new functions, the equilibrium differential equation changes to

$$\alpha(\xi)\frac{d^2}{d\xi^2}y(\xi) + \beta(\xi)\frac{d}{d\xi}y(\xi) + \gamma(\xi)\,y(\xi) - \int_0^\xi y(\xi)N(\xi)d\xi = \kappa(\xi)V_1 + \frac{M_1}{L} - V_1\xi \tag{12}$$

In which

$$\alpha(\xi)=\frac{E\,I(\xi)}{L^3}\left(\frac{N_1}{G\,A_w(\xi)}-1\right) \quad (13a)$$

$$\beta(\xi)=\frac{E\,I(\xi)}{L^2}\left(-\frac{\left(\frac{d}{d\xi}A_w(\xi)\right)N_1}{L\,G\,(A_w(\xi))^2}+\frac{N(\xi)}{G\,A_w(\xi)}\right) \quad (13b)$$

$$\gamma(\xi)=\frac{E\,I(\xi)}{L}\left(-\frac{\left(\frac{d}{d\xi}A_w(\xi)\right)N(\xi)}{L\,G\,(A_w(\xi))^2}+\frac{\frac{d}{d\xi}N(\xi)}{L\,G\,A_w(\xi)}-\frac{N_1}{E\,I(\xi)}\right) \quad (13c)$$

$$\kappa(\xi)=-\frac{E\,I(\xi)}{L^2\,G\,(A_w(\xi))^2}\left(\frac{d}{d\xi}A_w(\xi)\right) \quad (13d)$$

III. Differential Equation Solving by Using the Chebyshev Polynomials Approach

The functions $\alpha(\xi),\beta(\xi),\gamma(\xi)$ and $\kappa(\xi)$ can be expressed as

$$\alpha(\xi)=\sum_{i=0}^{CPD}\alpha_i\xi^i \quad (14a)$$
$$\beta(\xi)=\sum_{i=0}^{CPD}\beta_i\xi^i \quad (14b)$$
$$\gamma(\xi)=\sum_{i=0}^{CPD}\gamma_i\xi^i \quad (14c)$$
$$\kappa(\xi)=\sum_{i=0}^{CPD}\kappa_i\xi^i \quad (14d)$$

and the general solution of deflection $(y(\xi))$ can be expressed as

$$y(\xi)=\sum_{i=0}^{CPD}y_i\xi^i \quad (15a)$$

too.

Differentiating with respect to ξ, yields

$$y'(\xi)=\sum_{i=1}^{CPD}i\,y_i\,\xi^{i-1} \quad (15b)$$
$$y"(\xi)=\sum_{i=2}^{CPD}i\,(i+1)y_i\,\xi^{i-2} \quad (15c)$$

By substituting (14a)-(15c) into (12) we have

$$\sum_{n=0}^{CPD}\left[\sum_{i=0}^{n}\alpha_i(n+2-i)(n+1-i)y_{n+2-i}\right]\xi^n+\sum_{n=0}^{CPD}\left[\sum_{i=0}^{n}\beta_i(n+1-i)y_{n+1-i}\right]\xi^n+\sum_{n=0}^{CPD}\left(\sum_{i=0}^{n}\gamma_i\,y_{n-i}\right)\xi^n+\sum_{n=1}^{CPD}\frac{N_0\,y_{n-1}}{n}\xi^n=V_1\sum_{n=0}^{CPD}\kappa_n\xi^n+\frac{M_1}{L}-V_1\xi \quad (16)$$

Since (16) must be satisfied regardless of ξ, setting the coefficient of ξ^n to zero for $n=0,1,2,3,\dots$ we obtain three equations.

For $n=0$

$$\frac{M_1}{L}=2\alpha_0.y_2+\beta_0.y_1+\gamma_0.y_0-V_1.\kappa_0 \quad (17a)$$

For $n=1$

$$6\alpha_0.y_3+2\alpha_1.y_2+2\beta_0.y_2+\beta_1.y_1+\gamma_0.y_1+\gamma_1.y_0-N_0.y_0=V_1.\kappa_1-V_1 \quad (17b)$$

And For $n>1$

$$\sum_{i=0}^{n}\alpha_i(n+2-i)(n+1-i)y_{n+2-i}+\sum_{i=0}^{n}\beta_i(n+1-i)y_{n+1-i}+\sum_{i=0}^{n}\gamma_i\,y_{n-i}+\frac{N_0\,y_{n-1}}{n}=V_1\kappa_n \quad (17c)$$

$\Rightarrow$

$$y_{n+2}=\frac{1}{\alpha_0(n+2)(n+1)}*\left[-\sum_{i=0}^{n}\beta_i\,(n+1-i)y_{n+1-i}-\sum_{i=0}^{n}\gamma_i\,y_{n-i}+V_1\kappa X_n-\sum_{i=1}^{n}\alpha_i\,(n+2-i)(n+1-i)y_{n+2-i}-\frac{N_0\,y_{n-1}}{n}\right] \quad (17d)$$

In (17d) we can see that the term y_{n+2} is a linear function of y_0, y_1, y_2, y_3 and V_1. Series y_n may be found when the values of these terms are obtained.

The boundary conditions is

For $\xi=0$

$$y_0=0 \quad (18a)$$
$$y_1=\frac{L}{1-\frac{N_1}{G\cdot Aw(0)}}\left(\theta_1-\frac{V_1}{G\cdot Aw(0)}\right) \quad (18b)$$

And for $\xi=1$

$$y(1)=\sum_{i=0}^{CPD}y_i=\delta_2-\delta_1 \quad (19a)$$
$$y'(1)=\sum_{i=0}^{CPD}i\,y_i=\frac{L}{1-\frac{N_1}{G.Aw(1)}}\left(\theta_2-\frac{V_1-(\delta_2-\delta_1)\,N_0}{G.Aw(1)}\right) \quad (19b)$$

From the above we know that the series y_n may be found when the values of terms y_0, y_1, y_2, y_3 and V_1 are obtained. So we can rewrite the (19a) and (19b) as

$$c_1.y_1+c_2.y_2+c_3.y_3+c_4.V_1=(\delta_2-\delta_1) \quad (20a)$$
$$c_5.y_1+c_6.y_2+c_7.y_3+c_8.V_1=\frac{L}{1-\frac{N_1}{G.Aw(1)}}\left(\theta_2-\frac{V_1-(\delta_2-\delta_1)\,N(1)}{G.Aw(1)}\right) \quad (20b)$$

The deflection function $(y(\xi))$ is a general solution and it should be correct for all situations. We can assume that the values of y_2, y_3 and V_1 are equal to zero and just we have $y_1=1$. So the (20a) and (20b) will be reduce to

$$c_1=\sum_{i=0}^{CPD}y_i \quad (21a)$$
$$c_5=\sum_{i=0}^{CPD}i\,y_i \quad (21b)$$

(21a) and (21b) obtains the value of c_1 and c_5, simultaneously. By the same procedure and assuming a different values for terms y_1, y_2, y_3 and V_1 we can obtain the other coefficients. In Table II the assumed valued are shown.

TABLE II
Assumed Values for Calculating d_i

The terms that should be calculated	Assumed values for			
	V_1	y_3	y_2	y_1
c_1, c_5	0	0	0	1
c_2, c_6	0	0	1	0
c_3, c_7	0	1	0	0
c_4, c_8	1	0	0	0

The equilibrium conditions are

$$V_1 + V_2 = 0 \quad (22a)$$

$$V_1 L - N_1(\delta_2 - \delta_1) - L\int_0^1 y(\xi)N(\xi)d\xi - M_1 - M_2 = 0 \quad (22b)$$

For simplifying the term $\int_0^1 y(\xi)N(\xi)d\xi$ in (22b) it can be shown that

$$\int_0^1 y(\xi)N(\xi)d\xi = \sum_{i=0}\left(y_i\left[\sum_{j=0} N_j \frac{1}{i+j+1}\right]\xi^{i+j+1}\right)_{\xi=1} = \sum_{i=0}\left(y_i\left[\sum_{j=0} N_j \frac{1}{i+j+1}\right]\right) \quad (23)$$

And if we assume that the distributed axial force is constant ($N(\xi) = N_0$), we will have

$$\sum_{i=0}\left(y_i\left[\sum_{j=0} N_j \frac{1}{i+j+1}\right]\right) = \sum_{n=0}^{CPD}\frac{1}{n+1} y_n N_0 = d_1.y_1 + d_2.y_2 + d_3.y_3 + d_4.V_1 \quad (24)$$

In Which

$$d_j = \sum_{i=0}^{CPD}\frac{N_0}{i+1} y_i \;\; (j = 1, \ldots, 4) \quad (25)$$

TABLE III
ASSUMED VALUES FOR CALCULATION OF d_i

The term that should be calculated	Assumed values V_1	y_3	y_2	y_1
d_1	0	0	0	1
d_2	0	0	1	0
d_3	0	1	0	0
d_4	1	0	0	0

If the boundary deformations of the element $\delta_1, \theta_1, \delta_2$ *and* θ_2 are known, the above equations can be combined for solving V_1 and M_1. Assume that the matrix form of stiffness equation of the element is

$$\breve{K}\begin{Bmatrix}\delta_1\\ \theta_1\\ \delta_2\\ \theta_2\end{Bmatrix} = \begin{bmatrix}V_1\\ M_1\\ V_2\\ M_2\end{bmatrix} \quad (26)$$

Then the stiffness matrix would be

$$\breve{K} = \begin{bmatrix}-\psi_{15} & \psi_{13} & \psi_{15} & \psi_{14}\\ -\psi_{18} & \psi_{16} & \psi_{18} & \psi_{17}\\ \psi_{15} & -\psi_{13} & -\psi_{15} & -\psi_{14}\\ \psi_{19} & \psi_{20} & \psi_{21} & \psi_{22}\end{bmatrix} \quad (27)$$

The expression of ψ_i $(i = 1, \ldots, 22)$ are given in the Appendix A.

In theory, the approach described above is accurate. The unique possible error comes computationally from the representation of the realistic deflection y(z) and the defined function by Chebyshev Polynomial with definite terms, which affects directly no more than the coefficients c_i and d_i.

IV. VERIFICATION MODELS

A. Buckling Load

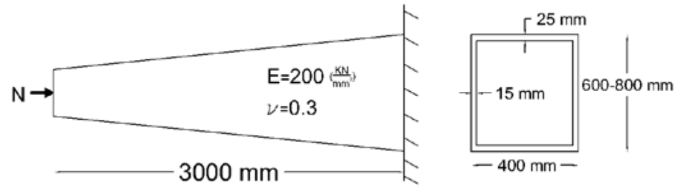

Fig. 2 Example-A Elastic buckling Load

The elastic buckling load of the tapered cantilever column shown in Fig. 2 is obtained by general structural analysis software SAP2000-14 and the procedure presented in this paper assuming local and out-of-plane buckling are prevented.

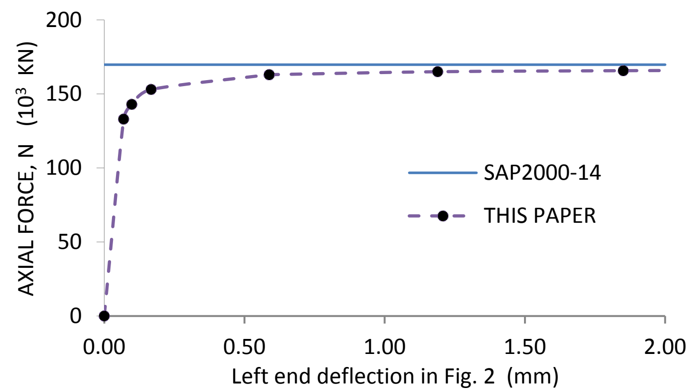

Fig. 3 P-Δ graph for example-A

Table IV summarizes these comparison results.

TABLE IV
EXAMPLE-A RESULTS COMPARISON

Technique	Elastic buckling load (KN)
SAP2000-14	169792
THIS PAPER	166900
Difference	1.7 %

B. Lateral deflection verification

The results in Table. V represents the comparison between lateral mid-span deflection of the beam-column shown in Fig. 4 calculated by utilizing this paper procedure and FEM.

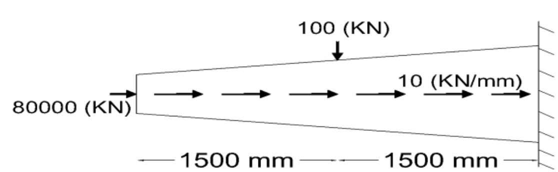

Fig. 4 Applied Forces in example-B

The geometrical and material properties of this beam is look like the beam assumed in Fig. 4. The results of this paper have been calculated by dividing the whole model into two elements. The FEM results have been calculated by general structural analysis software SAP2000-14 and utilizing the step representation method. SAP2000-14 considers just the shear deformation influences on stiffness matrix.

TABLE V
EXAMPLE-B RESULTS COMPARISON

Technique	Left end deflection ,δ1 (mm)
SAP2000-14	0.9462
THIS PAPER	0.9042
Difference	4.6 %

V. CONCLUSION

Axial Force and shear deformation influence the tapered beams stiffness-equation and significantly reduce their stiffness.

A tapered Timoshenko-Euler beam element (general form of beams) under a concentrated and distributed axial force with box-shaped cross-section is considered and governing equilibrium differential equation established. By utilizing the Chebyshev Polynomial approach technique, the governing differential equation of latticed and arbitrary variational cross-section members is fully solved and stiffness matrix established.

Like [1], The verifications proves that the proposed element and procedure could be successfully applied to general 2D structures.

APPENDIX

$\psi_1 = \frac{LGA_w(0)}{GA_w(0)-N_1}$ $\psi_2 = \frac{-L}{GA_w(0)-N_1}$ $\psi_3 = \frac{-\beta_0\,\psi_1}{2\alpha_0}$,

$\psi_4 = \frac{-\beta_0\,\psi_2+\kappa_0}{2\alpha_0}$ $\psi_5 = \frac{1}{2\,L\,\alpha_0}$,

$\psi_6 = \frac{\kappa_1-1-\gamma_0\psi_2-2\alpha_1\psi_4-2\beta_0\psi_4-\beta_1\psi_2}{6\alpha_0}$ $\psi_7 = \frac{-\psi_5(\alpha_1+\beta_0)}{3\alpha_0}$,

$\psi_8 = \frac{-\gamma_0\psi_1-2\alpha_1\psi_3-2\beta_0\psi_3-\beta_1\psi_1}{6\alpha_0}$,

$\psi_9 = \frac{LGA_w(1)}{GA_w(1)-N_1}$ $\psi_{10} = \frac{-L}{GA_w(1)-N_1}$

$\psi_{11} = \frac{L\,N_0}{GA_w(1)-N_1}$

$\psi_{12} = c_1\psi_2c_7\psi_7 + c_1\psi_2c_6\psi_5 + c_2\psi_4c_7\psi_7 + c_3\psi_6c_6\psi_5 + c_4c_7\psi_7 + c_4c_6\psi_5 - c_3\psi_7c_5\psi_2 + c_3\psi_7\psi_{10} - c_3\psi_7c_8 - c_3\psi_7c_6\psi_4 - c_2\psi_5c_5\psi_2 + c_2\psi_5\psi_{10} - c_2\psi_5c_7\psi_6 - c_2\psi_5c_8$,

$\psi_{13} = \frac{-1}{\psi_{12}}(c_7\psi_7c_1\psi_1 + c_7\psi_7c_2\psi_3 + c_6\psi_5c_1\psi_1 + c_6\psi_5c_3\psi_8 - c_3\psi_7c_5\psi_1 - c_3\psi_7c_6\psi_3 - c_2\psi_5c_5\psi_1 - c_2\psi_5c_7\psi_8)$

$\psi_{14} = \frac{(-c_3\psi_7-c_2\psi_5)\psi_9}{\psi_{12}}$

$\psi_{15} = \frac{c_7\psi_7+c_6\psi_5-\psi_{11}c_3\psi_7-\psi_{11}c_2\psi_5}{\psi_{12}}$

$\psi_{16} = \frac{1}{\psi_{12}}(c_5\psi_2c_2\psi_3 + c_5\psi_2c_3\psi_8 + c_7\psi_6c_1\psi_1 + c_7\psi_6c_2\psi_3 + c_6\psi_4c_1\psi_1 + c_6\psi_4c_3\psi_8 - c_1\psi_2c_6\psi_3 - c_1\psi_2c_7\psi_8 - c_2\psi_4c_5\psi_1 - c_2\psi_4c_7\psi_8 - c_3\psi_6c_5\psi_1 - c_3\psi_6c_6\psi_3 + c_8c_2\psi_3 - \psi_{10}c_2\psi_3 - c_4c_7\psi_8 + c_8c_3\psi_8 - \psi_{10}c_1\psi_1 - c_4c_5\psi_1 + c_8c_1\psi_1 - \psi_{10}c_3\psi_8 - c_4c_6\psi_3)$

$\psi_{17} = \frac{(c_1\psi_2+c_2\psi_4+c_3\psi_6+c_4)\psi_9}{\psi_{12}}$

$\psi_{18} = \frac{1}{\psi_{12}}(-c_5\psi_2 + \psi_{10} - c_7\psi_6 - c_8 - c_6\psi_4 + \psi_{11}c_2\psi_4 + \psi_{11}c_3\psi_6 + \psi_{11}c_4)$

$\psi_{19} = -L\psi_{15} - L(-d_1\psi_2\psi_{15} + d_2(-\psi_4\psi_{15} - \psi_5\psi_{18}) + d_3(-\psi_6\psi_{15} - \psi_7\psi_{18}) - d_4\psi_{15}) + \psi_{18} + N_1$

$\psi_{20} = L\psi_{13} - \psi_{16} - L(d_1(\psi_1 + \psi_2\psi_{13}) + d_2(\psi_3 + \psi_4\psi_{13} + \psi_5\psi_{16}) + d_3(\psi_6\psi_{13} + \psi_7\psi_{16} + \psi_8) + d_4\psi_{13})$

$\psi_{21} = L\psi_{15} - L(d_1\psi_2\psi_{15} + d_2(\psi_4\psi_{15} + \psi_5\psi_{18}) + d_3(\psi_6\psi_{15} + \psi_7\psi_{18}) + d_4\psi_{15}) - N_1 - \psi_{18}$

$\psi_{22} = L\psi_{14} - L(d_1\psi_2\psi_{14} + d_2(\psi_4\psi_{14} + \psi_5\psi_{17}) + d_3(\psi_6\psi_{14} + \psi_7\psi_{17}) + d_4\psi_{14}) - \psi_{17}$

$N_t = -N_2 = \mathrm{N}_1 + \int_0^{\mathrm{L}} \mathrm{N(z)dz} = \mathrm{N}_1 + \mathrm{N}_0\mathrm{L}$

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