Closed-form linear stability conditions for Magnetoconvection in a sparsely packed porous medium

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Abstract—The linear stability analysis of magnetoconvection in a sparsely packed porous medium with free-free boundary conditions has been thoroughly investigated by Chandrasekhar [4], who determined the marginal stability boundary and critical wave numbers for the onset of convection and over stability as a function of the Chandrasekhar number Q. No closed-form formulae appeared to exist and the results were tabulated numerically. However, by taking the Rayleigh number R as an independent variable it is found that the expressions are simplified remarkably when the condition is $(Pr_2/\mathrm{Pr}_1) < 1$. The marginal stability boundary is described by the curve

$$Q = Q_{sc} = \frac{R}{\pi^2}\left[1-\left(\frac{\Lambda R_{rb}}{R}\right)^{\frac{1}{3}}\right]$$

where $R_{rb} = (27/4)\pi^4$ 4 is critical Rayleigh number for the onset of stationary convection in a non-magnetic system. For $(Pr_2/\mathrm{Pr}_1) > 1$, the marginal stability boundary is determined by this curve until it is intersected by the curve

$$Q = Q_{oc} = \frac{RB_2}{\pi^2 B_1}\left[1-\left(\frac{\Lambda R_{rb}}{RB_2}\right)^{\frac{1}{3}}\right].$$

Where

$$B_1 = \frac{M^2\,\mathrm{Pr}_1^2}{\left(M\,\mathrm{Pr}_1+\phi\,\mathrm{Pr}_2\right)\left(1+M\Lambda\phi\,\mathrm{Pr}_1\right)},\; B_2 = \frac{\phi^3\,\mathrm{Pr}_2^2}{\left(M\,\mathrm{Pr}_1+\phi\,\mathrm{Pr}_2\right)\left(1+M\Lambda\phi^2\,\mathrm{Pr}_1\right)}.$$

Keywords**—**Bifurcation points, Boussinesq approximation. Convection.

I. Introduction

The original motivation for studying the effects of magnetic field on Rayleigh Benard convection arose from the attempts to explain the origin of sunspots and starspots. Sunspots and starspots are dark because thermal convection is suppressed by a strong magnetic field, which allows only weaker time dependent motion. Thermal convection takes place in an electrically conducting fluid of planetory or stellar dimension only in the presence of a strong magnetic field. The existence of the sunspots and starspots motivated theoretical studies of convection in strong magnetic field within the framework of the Boussinesq approximation [4]. Magnetoconvection exhibits a particularly rich variety of behavior when the ratio of the magnetic to the thermal diffusivity is small [7].

Magnetoconvection in an electrically conducting fluid in a nonporous medium has been studied extensively [3, 5, 8, and 10]. However, magnetoconvection in a porous medium has received limited attention inspite of its application in geophysical fluid dynamics problems. Rudraiah et al., [9] have studied double diffusive convection in porous medium by considering Darcy flow model which is relevant to densely packed flow permeability porous media and cannot account for inertia and boundary effects. Palm et al., [6] investigated Rayleigh-Benard convection problem in a porous medium. The Rayleigh-Benard convection has been extensively used to study the motion in the earth's interior and to understand the mechanism of transfer of energy from the deep interior of the earth to the shallow depths [11]. This transfer of energy plays an important role in geothermal activities. Benerji babu et al. [2] have investigated the problem of nonlinear rotating magnetoconvection in a sparsely packed porous medium.

In this paper we investigate the problem of closed-form linear stability conditions for magnetoconvection in a sparsely packed porous medium with free-free boundary conditions. In section II , we discuss the linear stability analysis by taking Rayleigh number R as an independent variable. In section III, we discuss the limit of zero magnetic diffusivity. In section IV, we conclude the findings of the paper.

II. Linear Stability Analysis

Consider an electrically and thermally conducting fluid between two conducting parallel plates of length infinitely separated by distance d. A uniform magnetic field H_o is applied in the vertical z-direction. The lower plate is heated from below, the upper and lower plates are assumed to be stress-free. Physical properties of the fluid are assumed to be constant, except for the density in the buoyancy term, so that the Boussinesq approximation is valid. The temperature difference across the stress-free boundaries is $\Delta T'$ and the flow in the sparsely packed porous medium is governed by the Darcy-Lapwood-Brinkman model. The relevant basic equations are obtained from Benerji babu et al. [1]. Procedure to analyze stability properties of the system have been discussed in detail [1]. The same procedure has been utilized herein.

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The expression (27) in [1] is simplified by assuming the condition $W = \sin \pi z,$ to get a third degree polynomial p of following form:

$$p^3 + Bp^2 + Cp + D = 0, \tag{1}$$

where

$$B = \left(1 + M\Lambda\phi \Pr_1 + \frac{M \Pr_1}{\phi \Pr_2}\right)\delta^2 + \frac{M\phi \Pr_1}{D_a}, \tag{2}$$

$$C = \left(M\Lambda\phi \Pr_1 + \frac{\Lambda M^2 \Pr_1^2}{\Pr_2} + \frac{M \Pr_1}{\phi \Pr_2}\right)\delta^4 + \left(\frac{M^2 \Pr_1^2}{D_a \Pr_2} + \frac{M\phi \Pr_1}{D_a}\right)\delta^2 + \frac{Q\pi^2 M^2 \phi \Pr_1^2}{pr_2} - \frac{Rq^2 M\phi \Pr_1}{\delta^2}, \tag{3}$$

$$D = \frac{\Pr_1^2}{\Pr_2}\left(\Lambda M^2\delta^6 + \frac{M^2\delta^4}{D_a} + Q\pi^2 M^2\delta^2 - Rq^2 M^2\right). \tag{4}$$

The dimensionless parameters required for the description of the motion are: Rayleigh number $R = g\alpha\Delta T d^3/\kappa\upsilon,$ thermal Prandtl number $\Pr_1 = \upsilon/\eta,$ magnetic Prandtl number $\Pr_2 = \upsilon/\eta,$ Chandrasekhar number $Q = \mu_m H_o^2 d^2/4\pi\rho_0\upsilon\eta$ and Darcy number $D_a = \kappa/d^2.$ Setting $p = i\omega$ in equation (1), and considering its real and imaginary parts we get

$$D - B\omega^2 = 0,$$

$$\omega^2 - C = 0.$$

a. Stationary Convection ($\omega = 0$)

Substituting $\omega = 0$ into equation (1), we get $D = 0,$ which can be written as

$$\Lambda\left(r+\pi^2\right)^3 + \frac{\left(r+\pi^2\right)^2}{D_a} + Q\pi^2\left(r+\pi^2\right) - Rr = 0, \tag{5}$$

Where $r = q^2.$

For a fixed equation (5), determines a curve given as

$$Rr = \left(r+\pi^2\right)\left[\Lambda\left(r+\pi^2\right)^2 + \frac{\left(r+\pi^2\right)}{D_a} + Q\pi^2\right]. \tag{6}$$

In (R, r)-plane $R > R_{sc}$ (critical Rayleigh number for the onset of stationary convection) there are two regions, when $D > 0$ the system is stable and Unstable for $D < 0$. If $D = 0$, plot follows a curve as shown in Fig. 1. Analytic expression (34) in [1] has been used to find critical Rayleigh number by considering R as a dependent variable. Similarly we can find an analytical expression for Chandrasekhar number by considering R as an independent variable [5]. This critical Chandrasekhar number is computed as follows:

The equation (6) can be written as

$$\frac{Rr}{\left(r+\pi^2\right)} = \left[\Lambda\left(r+\pi^2\right)^2 + \frac{\left(r+\pi^2\right)}{D_a} + Q\pi^2\right]. \tag{7}$$

The derivative of equation (7) with respect to r gives

$$R = \frac{2\Lambda\left(r+\pi^2\right)^3}{\pi^2} + \frac{\left(r+\pi^2\right)^2}{D_a}. \tag{8}$$

On substituting equation (8) into equation (5), we get

$$2\Lambda\left(\frac{r}{\pi^2}\right)^3 + \left(3\Lambda + \frac{1}{\pi^2 D_a}\right)\left(\frac{r}{\pi^2}\right)^2 = \Lambda + \frac{Q}{\pi^2} + \frac{1}{\pi^2 D_a}. \tag{9}$$

Equation (9) is nothing but equation (33) in [1] at $r = q^2$ and $q = q_{sc}$. We can write equation (8) in terms of r as

$$r = \left(\frac{R\pi^2}{2\Lambda}\right)^{\frac{1}{3}} - \pi^2 \text{ or } q_{sc} = \left[\left(\frac{R\pi^2}{2\Lambda}\right)^{\frac{1}{3}} - \pi^2\right]^{\frac{1}{2}}, \tag{10}$$

Where $Da \to \infty$. From equation (10) we consider only positive values of r, since $r = q^2$ (> 0). Substituting equation (10) into equation (5), we get critical Chandrasekhar number $Q = Q_{sc}(R)$, where

$$Q = Q_{sc} = \frac{R}{\pi^2}\left[1 - \left(\frac{\Lambda R_{rb}}{R}\right)^{\frac{1}{3}}\right] \text{ where } R_{rb} = \frac{27\pi^4}{4}. \tag{11}$$

Here R_{rb} is the critical Rayleigh number for the onset of stationary convection of Rayleigh-Benard convection in the absence of magnetic field. Fig. 2 is plotted in (R, Q)-plane for the expression (11). In this figure $Q = 0$ on the R-axis. From R-axis, the curve (11) starts from $R = R_{rb}$. In (R, Q)-plane we check the sign of D in a range of R with $q = q_{sc}(R)$ at a fixed $Q = Q_{sc}$. For the values $\{R,Q\}$ which are on the left of the curve (11), $D > 0$ and $D < 0$ for the values $\{R, Q\}$ which are to the right of the curve (11) and $D = 0$ on the curve (11).

b. *Oscillatory Convection (* $\omega^2 > 0$ *)*

The condition D > 0 is not enough to discuss the stability of the system. We should also to check the sign of $BC - D$ along with the sign of D for the stability of the system. Thus $BC - D = 0$ gives

$$\Lambda\left(r+\pi^2\right)^3 + Q\pi^2 B_1\left(r+\pi^2\right) - RrB_2 = 0, \tag{12}$$

or

$$R_o = \frac{\delta_o^2}{B_2 q_o^2}\left[\Lambda\delta_o^4 + Q\pi^2 B_1\right], \tag{13}$$

Where R_o is the Rayleigh number for oscillatory convection. The frequency for the oscillations is given by $\omega^2 =$C or $\omega^2 =$D/B (since R is an independent variable). If R is a dependent variable then we get ω^2 by eliminating R from

ω^2=C and ω^2=D/B which is given as

$$\omega^2 = \frac{M^2 \Pr_1^2}{\phi^2 \Pr_2^2 (1 + M\Lambda\phi \Pr_1)} \left[Q\pi^2\phi\left(\phi \Pr_2 - M \Pr_1\right) - \delta_o^4 (1 + M\Lambda\phi \Pr_1) \right]. \tag{14}$$

We follow similar procedure to compute analytical expressions of critical Chandrasekhar number Q_{oc} and critical wave number q_{oc} for oscillatory convection as we have obtained Q_{sc} and q_{sc} for stationary convection. Here we compute the ana-
lytical expressions Q_{oc} and q_{oc} directly by comparing the equations (5) and (12) we obtain

$$Q = QB_1 \text{ and } R = RB_2.$$

Substituting above conditions into equations (10) and (11), we get

$$q = q_{oc}\left(R, \Pr_1, \Pr_2, \Lambda, \phi, M\right) = \left[\left(\frac{RB_2\pi^2}{2} \right)^{\frac{1}{3}} - \pi^2 \right]^{\frac{1}{2}}, \tag{15}$$

and

$$Q = Q_{oc}\left(R, \Pr_1, \Pr_2, \Lambda, \phi, M\right) = \frac{RB_2}{\pi^2 B_1} \left[1 - \left(\frac{\Lambda R_{rb}}{RB_2} \right)^{\frac{1}{3}} \right]. \tag{16}$$

In the (R, Q)-plane, on the R-axis $Q = 0$ and the curve corresponds to (16) which always starts from $R = R_{rb}/B2$. In this plane $BC - D > 0$ for the values $\{R, Q\}$ which is the left of the curve (16) and $BC - D < 0$ for the values $\{R, Q\}$ which are to the right of the curve (16). When $BC - D > 0$ and $D > 0$ we get one damped mode and two oscillatory modes. Thus the system is stable in $BC - D > 0$ and $D > 0$ region. In Figs. 3a-3d and 4a-4d, solid and dotted lines intersect at

$$R = R_{ct} = \Lambda\Upsilon R_{rb}, Q = Q_{ct} = \frac{\Lambda R_{rb} \Upsilon^{\frac{2}{3}}}{\pi^2} \left(\Upsilon^{\frac{2}{3}} - 1 \right),$$

Where

$$\Upsilon = \frac{\left(1 - K_1 B_2^{-\frac{1}{3}}\right)}{\left(1 - K_1\right)^3} \text{ and } K = \left(\frac{\Pr_2}{\Pr_1} \right)^2 \frac{\left(1 + M\Lambda\phi \Pr_1\right)\phi^3}{\left(1 + \Lambda\phi^2 \Pr_2\right) M^2}. \tag{17}$$

The suffix ct in equation (17) stands for parameter at co-dimension two bifurcation point. The Rayleigh number $R = R_{ct}$ is obtained by eliminating Q using equation (11) and (16). By substituting $R = R_{ct}$ either equation (11) or into equation (16), we get $Q = Q_{ct}$. At Q_{ct}, $Q_{sc} = Q_{oc}$ and $q_{sc} \neq q_{oc}$. For $Pr_2 \leq Pr_1$, the curves Qsc and Qoc never intersect at any point in (R,Q)-plane. This implies that we do not get oscillatory convection for $Pr_2 \leq Pr_1$. In Figs. 3a-3d Pr_1 increases for fixed Pr_2 and the intersection point $\{R_{ct}, Q_{ct}\}$ moves upward. In Figs. 4a-4d the intersection point $\{R_{ct}, Q_{ct}\}$ moves downward for a fixed ratio $Pr_2/Pr_1 > 1$. In the above Figs. 3a-3d and 4a-4d, the frequency ω^2 changes its sign from negative to positive before the intersection point as R increases. When $Q < Q_{ct}$ we get stationary convection as a first instability, while for $Q > Q_{ct}$ the first instability will be oscillatory convection.

Eliminating Q from equations $C = 0$ and $D = 0$, we get

$$q^6 + 3q^4\pi^2 + \left[3\pi^4 + \frac{R\left(M\phi \Pr_1 - \phi^2 \Pr_2\right)}{1 + \Lambda\phi^2 \Pr_2} \right] q^2 + \pi^6 = 0. \tag{18}$$

Equation (18), gives two Takens-Bogdanov bifurcation points for $Pr_2 > Pr_1$ (Fig. 5). We do not get oscillatory convection for $Pr_2 \leq Pr_1$, since equation (18) does not give positive roots.

III. In the Limit of Zero Magnetic Diffusivity

In the limit $\eta \to 0$, $Pr_2 \to \infty$ and $Pr_2/Pr_1 >> 1$, the stability properties of the system with $\eta \to 0$ can be determined by letting $Pr_2 \to \infty$ and $Q \to \infty$ such that Q/Pr_2 is finite in the coefficients B,C,D given by (2)-(4).

$$B = \left(1 + M\Lambda\phi \Pr_1\right)\delta^2 + \frac{M\phi \Pr_1}{D_a},$$

$$C = M\Lambda\phi \Pr_1 \delta^4 + \frac{M\phi \Pr_1}{D_a}\delta^2 + \pi^2 M^2 \Pr_1^2 \left(\frac{Q}{\Pr_2} \right) - \frac{Rq^2 M\phi \Pr}{\delta^2},$$

$$D = \pi^2 M^2 \Pr_1^2 \delta^2 \left(\frac{Q}{\Pr_2} \right).$$

If $Pr_2/Pr_1 >> 1$ and η finite but small, we will get only oscillatory convection. If $\eta << 1$ then double-diffusive system will behave like a single diffusive system with finite Q/Pr_2 and we will get only stationary convection.

IV. Conclusion

We have revisited the linear problem of magnetoconvection in a sparsely packed porous medium using Earth's outer core by stress-free boundary conditions. Even though free-free boundary conditions cannot be achieved in laboratory, one can use it in geophysical fluid dynamic applications to Earth's outer core since they allow simple trigonometric eigenfunctions. Our goal is to identify the region of parameter values, for which rolls emerge at the onset of convection. Chandrasekhar [4] described the stationary convection and oscillatory convection curves in the RQ- plane as curves $R_s(Q)$ and $R_o(Q, Pr_1, Pr_2, \Lambda, \phi, M)$ respectively. The critical wave number for stationary convection and overstability (oscillatory convection), $q_{sc}(Q)$ and $q_{oc}(Q, Pr_1, Pr_2, \Lambda, \phi, M)$ and the oscillation frequency of the oscillatory convection ω $(Q, Pr_1, Pr_2, \Lambda, \phi, M)$ were also calculated by him. No closed formulae for the curves defining the marginal stability boundary were noted by him nor any for the critical number.

By considering Rayleigh number R as the independent variable, we found in section II simple expression for the stationary convection curve $Q_{sc}(R)$ and for the oscillatory convection curve $Q_{oc}(R, Pr_1, Pr_2, \Lambda, \phi, M)$ given by equations (11) and (16). Similarly, the critical horizontal wave number for the onset of stationary convection $qsc(R)$ and the onset of oscillatory convection $q_{oc}(R, Pr_1, Pr_2, \Lambda, \phi, M)$ are given by the cures (10) and (15).We have obtained explicitly $Q = Q_{ct}(Pr_1, Pr_2, \Lambda, \phi, M)$ and $R = R_{ct}\ (Pr_1, Pr_2, \Lambda, \phi, M)$ viz. equation (17) for which we get co-dimension two bifurcation point.We have also obtained explicitly Takens-Bogadanov bifurcation point which is the intersection point of the neutral curves corresponding to stationary and oscillatory convection.

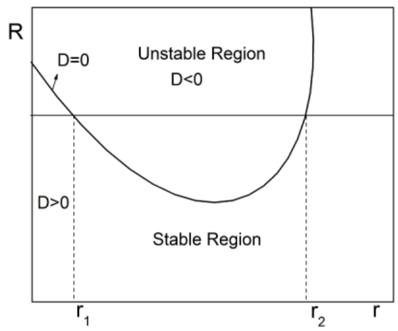

Fig. 1 A typical diagram showing the stability regions of the system for stationary convection.

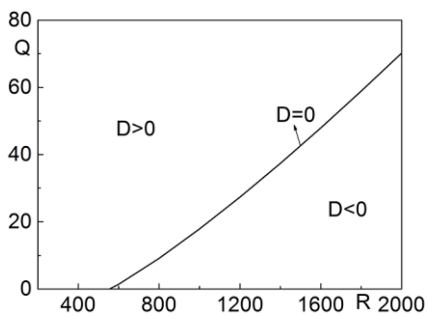

Fig. 2 For the curve (11), $D = 0$ with $q = q_{sc}$ given by equation (10). The system is stable for $D > 0$ and unstable for $D < 0$.

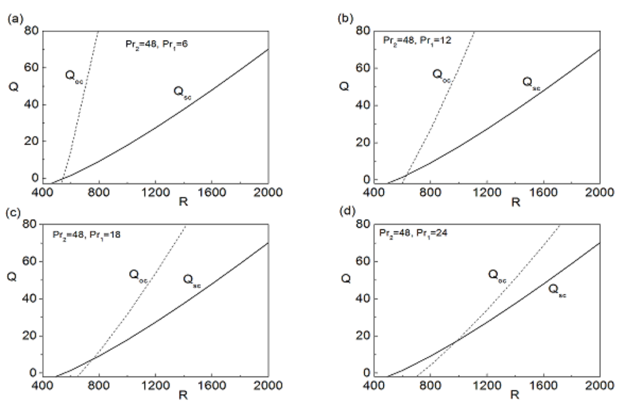

Fig. 3 In above figures each solid line starts from $R = Rrb$ and dotted line starts from $R = Rrb/B2$ on the R-axis. The intersection point of the solid and the dotted line gives co-dimension two bifurcation point. The frequency ω^2 changes its sign from negative to positive as R increases and ω^2 at co-dimension to bifurcation point.

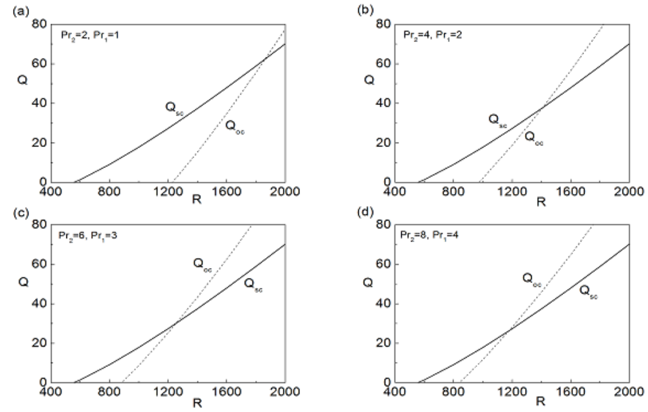

Fig.4 The same as Fig. 3 with fixed ration $Pr2/Pr1 = 2$

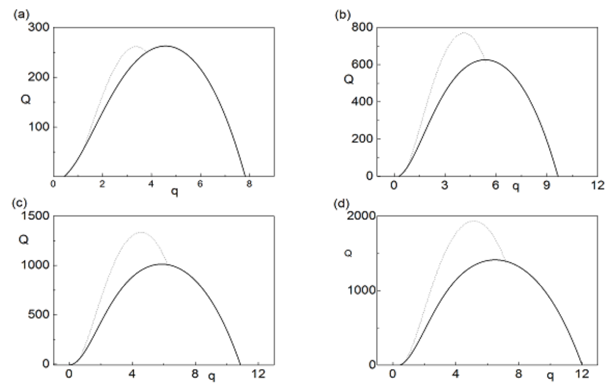

Fig. 5 on solid lines we have taken Chandrasekhar number for stationary convection and dotted lines we have taken Chandrasekhar number for oscillatory convection. These figures are plotted for the fixed values of $Da = 1500$, $\Lambda = 0.85$, $\phi = 0.9$, $Pr1 = 1$, $Pr2 = 1.5$ and $M = 0.9$. (a) $R = 5000$ (b) $R = 10000$ (c) $R = 15000$ (d) $R = 20000$.

References

[1] A. Benerji babu, Ravi Ragoju and S. G. Tagare, "Nonlinear Magnetoconvection in a Sparsely Packed Porous Medium", *International Journal of Geophysics*, vol 2011, pp. 1-17, Article ID 207123, 2011.

[2] A. Benerji babu, Ravi Ragoju and S. G. Tagare, "Nonlinear Thermohaline Magnetoconvection in a Sparsely Packed Porous Medium", *Journal of porous medium*, vol 17, pp. 31-57, 2014.

[3] H. R. Brand, P. C. Hohenberg and V. Steinberg, "Codimension-2 bifurcations for convection in binary fluid mixtures", *Physical Review A*, vol 30, pp. 2548-2561, 1984.

[4] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Dover publications, 1961.

[5] R. C. Kloosterziel and G. F. Carnevale, "Closed-form linear stability conditions for magnetoconvection", *Journal of Fluid Mechanics*, vol 490, pp. 333-344, 2003.

[6] E. Palm, J. E. Weber and O. Kvernvold, "On steady convection in a porous medium", *The Journal of Fluid Mechanics*, vol 64, pp. 153-161, 1972.

[7] M. R. E. Proctor and N. O. Weiss, "Magnetoconvection", *Reports on Progress in Physics*, Vol 45, pp. 1317-1379, 1982.

[8] P. H. Roberts and C. A. Jones, "The onset of magnetoconvection at large prandtl number in a rotating layer I. Finite magnetic diffusion", *Geophysical and Astrophysical Fluid Dynamics*, vol 92, pp. 289-325, 2000.

[9] N. Rudraiah, P. K. Shrimani and R. Friedrich, "Finite amplitude convection in a two-component fluid saturated porous layer", *Intl. Journal of Heat Mass Transfer*, vol 2, pp. 715-722, 1982.

[10] S. G. Tagare and Y. Rameshwar, "Magnetoconvection in rotating stars", *Astrophysics and Space Science*, vol 284, pp. 983-999, 2003.

[11] D. L. Turcotte and E. R. Oxburgh, "Mantle convection and the new global tectonics", *Annu. Rev. Fluid Mech.*, vol 4, pp. 33-68, 1972.