On the Sensitivity of Poisson EWMA Control Chart

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Abstract**—**Until recently, the application of the runs rules was restricted to the Shewhart-type control charts. In this paper, we propose an EWMA control chart with fast initial response (FIR) using $\mathbf{2/2}$ runs rule scheme for monitoring Poisson observations. Using Monte Carlo simulations, we studied and compared the average run length (ARL), the Average Ratio ARL (ARARL), the Average Extra Quadratic Loss (AEQL) and the Performance Comparison Index (PCI) of the proposed runs rule FIR Poisson EWMA chart with some other Poisson EWMA procedures. It is shown that the use of runs rule and FIR feature has substantially increased the sensitivity of Poisson EWMA control chart without decreasing the out-of-control ARL. The proposed chart is more effective in detecting moderate shifts in Poisson process than some of the existing procedures.

Keywords**—**Average run length, loss function, Poisson distribution, statistical process control.

I. Introduction

STATISTICAL process control (SPC) techniques are the statistical methods widely used to monitor and improve the quality and productivity of industrial processes and service operations. It primarily involves the implementation of control charts. Control charts are graphical tools used to monitor production process in order to distinguish special causes of variations that result in significant disturbances in the process performance from the omnipresent common causes of variation that are naturally present in process [1].

This technology of statistical quality control has been popularized in the worldwide and is extensively used to detect process changes to prevent the production of defective products. Among the many forms of process changes, process mean shifts are the primary focus of most quality control chart designs. In particular, the control chart mostly used in practice, the Shewhart-type control chart [1].

The practical experience in use of control charts indicated that the Shewhart charts are only very effective in the detection of larger shift of production process. When the small shift of process with a trend occurs, however, it is very difficult for Shewhart control charts to identify the drift of the process in time. For this reason, two alternative control charts, the exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) were introduce to particularly address smaller and moderate process shifts [1], [2].

Despite their lower levels of industrial use, the EWMA and the CUSUM control charts are statistically more effective in detecting small shifts than the Shewhart chart. As computing power increases and becomes easily accessible, the EWMA control chart in particular, is gaining its share of practical uses, especially in hi-tech industries where the tolerance level for process deviations is becoming extremely tight [3].

To increase the sensitivity of EWMA control charts in detecting wide range of process shifts, several design structures have been suggested in the literature. For example, [4] and [5] analyzed the performance of the fast initial response (FIR) feature while [6] examined the benefits of using runs rules schemes with EWMA charts. In this article, we use the combination of the FIR and runs rule with Poisson EWMA (PEWMA) control chart [7], [8] to monitor Poisson observations.

The outline of the rest of this article is as follows. In the next Section, we present the classical PEWMA control chart. In Section 3, we describe the design structure of the proposed PEWMA chart with FIR and runs rule. In Section 4, we compare the performance of different PEWMA control charts. Finally, we summarize our findings in Section 5.

II. The Classical Poisson EWMA Control Chart

In certain production processes, when the quality characteristic is the number of nonconformities per unit of measurement, Poisson distribution is often use as a model [8]. [7] and [8] studied the classical PEWMA control chart. Let $x_1, x_2, x_3, \ldots$ be independent and identically distributed Poisson random variables with mean μ. Assume that $x_i,\ i = 1, 2, 3, \ldots$ has an in-control mean $\mu = \mu_0$ and an out-of-control mean $\mu = \mu_1$. If μ_0 known, then the PEWMA control statistic z_i for monitoring the process mean is defined by

$$z_i = \lambda x_i + (1-\lambda) z_{i-1} \tag{1}$$

where λ is the smoothing constant with $0 < \lambda \leq 1$. If the process is in-control, the initial value of the statistic is set as $z_0 = \mu_0$. The variance of z_i is

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$$\text{var}(z_i) = \mu_0 \left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}]. \quad (2)$$

Thus, the lower and upper time varying control limits for PEWMA chart [8] are given respectively by

$$LCL = \mu_0 - L\sqrt{\frac{\lambda\mu_0}{2-\lambda}[1-(1-\lambda)^{2i}]} \quad (3)$$

and

$$UCL = \mu_0 + L\sqrt{\frac{\lambda\mu_0}{2-\lambda}[1-(1-\lambda)^{2i}]} \quad (4)$$

where L is the control limit constant. The PEWMA chart gives an out-of-control signal when $z_i < LCL$ or $z_i > UCL$.

Most often, the statistical performance of a control chart is measure by the average run length (ARL). ARL represents the average number of samples that must be plotted between the lower and upper control limits of a control chart before an out-of-control signal is observed [1]. To compare the performance of two or more control charts effectively, it is recommended to set a common in-control ARL for the charts. The chart with smaller out-of-control ARL is considered more superior and can detect process shifts more quickly than other control charts.

III. The Proposed Poisson EWMA Control Chart

The use of FIR feature in EWMA control charts have significantly improve the ARL values for the out-of-control processes but at the cost of early false alarm rate. In other words, the feature decreases both the out-of-control and the in-control ARL values of an EWMA chart [4], [5] and [6]. This problem of decrease in an in-control ARL may be overcome by addition of a runs rule scheme [6]. The application of runs rules in EWMA control chart and PEWMA in particular, are very few. In this article, we develop a 'simple *two-out-of-two* runs rule FIR PEWMA scheme'.

The design structure of the proposed scheme is defined as follows: a process is considered out-of-control if two consecutive points plots either below the LCL_{fir} or above the UCL_{fir}, where

$$LCL_{fir} = \mu_0 - F_{adj}L\sqrt{\frac{\lambda\mu_0}{2-\lambda}[1-(1-\lambda)^{2i}]} \quad (5)$$

and

$$UCL_{fir} = \mu_0 + F_{adj}L\sqrt{\frac{\lambda\mu_0}{2-\lambda}[1-(1-\lambda)^{2i}]} \quad (6)$$

where F_{adj} denotes the FIR [5] feature defined by:

$$F_{adj} = 1-(1-f)^{1+a(i-1)}. \quad (7)$$

Here, $a > 0$ is the adjustment parameter and f is the distance from the starting value with $0 < f \le 1$. Following [9] and [5], we have used $F_{adj} = 1-(0.5)^{1+0.3(i-1)}$, $i = 1,2,3,\ldots$ throughout this study. The ARL values of a PEWMA chart can be computed using Markov Chains approximations, integral equations or Monte Carlo simulations. In this article, we follow the later approach through an algorithm developed in R.

IV. Performance Comparison

The performance of the proposed scheme is compared to other approaches for monitoring Poisson output of a process. For better comparison, we set the in-control ARL_0 for all the control chart at 200. Using smoothing constants $\lambda = 0.05, 0.1, 0.2, 0.3, 0.4,$ and 0.5 with different combinations of L, we assume the process follows a Poisson distribution with $\mu_0 = 4$ (without loss of generality). We simulated 10,000 iterations for each mean shift using individual observation. In addition to the ARL, which measures the performance of a control chart at some specific levels, we implore the use of the following indices to measure the overall performance of the control charts [10], [11], [12]. These includes the Average Extra Quadratic Loss (AEQL),

$$AEQL = \frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}} \delta^2 ARL(\delta)\ \mathrm{d}\delta \quad (8)$$

the Average Ratio of ARL (ARARL)

$$ARARL = \frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}} \frac{ARL(\delta)}{ARL_{best}(\delta)}\ \mathrm{d}\delta \quad (9)$$

and the Performance Comparison Index (PCI)

$$PCI = \frac{AEQL}{AEQL_{best}} \quad (10)$$

where $\delta = \mu_1 - \mu_0$ is the mean shift in the Poisson process; $ARL_{best}(\delta)$ and $AEQL_{best}(\delta)$ are generated by the best chart. The results obtained for the increases in process mean are presented in Tables I–VI.

TABLE I
ARL COMPARISON OF POISSON EWMA CHARTS ($\mu_0 = 4$, $\lambda = 0.05$)

	Classical PEWMA	Double PEWMA	RR PEWMA	RR HS PEWMA	RR FIR PEWMA
μ	*L* =2.270	*L* = 1.705	*L* = 2.071	*L* = 2.055	*L* = 2.066
4.0	200.46	200.15	200.05	200.55	200.30
4.5	44.99	41.34	47.75	44.10	43.28
5.0	16.23	15.20	18.19	14.30	14.43
5.5	8.85	8.31	10.32	7.38	7.26
6.0	5.78	5.23	6.95	4.57	4.26
6.5	4.15	3.76	5.34	3.20	2.71
7.0	3.20	2.91	4.29	2.32	1.91
7.5	2.62	2.33	3.71	1.79	1.40
8.0	2.23	1.96	3.23	1.45	1.05
AQEL	24.222	21.977	31.201	18.451	16.128
PCI	1.502	1.363	1.935	1.144	1.000
ARARL	1.480	1.344	1.895	1.137	1.000

A. Proposed vs. the Classical PEWMA

The classical Poisson EWMA studied by [7], [8] and [13] has better out-of-control ARL performance than the Shewhart *c* chart. Comparing the simulated ARL values for the PEWMA with the time varying control limits with our scheme, indicates that the proposed chart have smaller ARL for all values of smoothing constant λ when mean shift is greater than 0.5 (cf. Tables I–VI). From the overall viewpoint, the performance of the proposed chart is almost as twice as the PEWMA chart in terms of the AEQL, PCI and ARARL. For example, the classical PEWMA is less effective than the proposed chart by 51.7% in terms of PCI when $\lambda = 0.1$, (cf. Table II).

TABLE II
ARL COMPARISON OF POISSON EWMA CHARTS ($\mu_0 = 4,\ \lambda = 0.1$)

	Classical PEWMA	Double PEWMA	RR PEWMA	RR HS PEWMA	RR FIR PEWMA
μ	*L =2.474*	*L =1.996*	*L =2.205*	*L =2.206*	*L =2.211*
4.0	200.37	200.02	200.26	200.40	200.57
4.5	49.82	47.20	53.03	50.02	49.44
5.0	18.17	16.90	20.07	16.42	16.16
5.5	9.82	9.10	11.16	7.98	7.63
6.0	6.20	5.88	7.51	4.93	4.56
6.5	4.48	4.10	5.64	3.40	2.84
7.0	3.45	3.13	4.55	2.52	2.02
7.5	2.79	2.51	3.83	1.91	1.48
8.0	2.32	2.11	3.38	1.55	1.10
AQEL	26.155	24.039	33.147	20.005	17.246
PCI	1.517	1.394	1.922	1.160	1.000
ARARL	1.498	1.378	1.891	1.154	1.000

TABLE III
ARL COMPARISON OF POISSON EWMA CHARTS ($\mu_0 = 4,\ \lambda = 0.2$)

	Classical PEWMA	Double PEWMA	RR PEWMA	RR HS PEWMA	RR FIR PEWMA
μ	*L =2.645*	*L =2.282*	*L =2.237*	*L =2.231*	*L =2.237*
4.0	200.00	200.21	200.54	200.58	200.66
4.5	55.66	54.26	60.09	57.14	57.19
5.0	21.29	19.58	22.90	20.03	18.82
5.5	11.15	10.27	12.26	9.36	8.64
6.0	7.12	6.59	8.03	5.58	4.80
6.5	5.05	4.73	5.89	3.68	3.01
7.0	3.91	3.59	4.67	2.61	2.10
7.5	3.13	2.90	3.93	2.05	1.54
8.0	2.65	2.43	3.41	1.60	1.15
AQEL	29.716	27.524	34.907	22.143	18.629
PCI	1.595	1.477	1.874	1.189	1.000
ARARL	1.587	1.471	1.864	1.186	1.000

B. Proposed vs. the Double PEWMA

The double Poisson EWMA studied by [13] is more sensitive to small process mean shifts than the classical PEWMA chart (cf. Tables I–VI). Comparison shows that the proposed chart has larger out-of-control ARL values than the double PEWMA in a very small region when the mean shift is $(0 < \delta \leq 1)$. Beyond this region, the proposed chart is more superior in detecting mean changes in Poisson process. In fact, the overall performance, in terms of the AEQL, PCI and ARARL, indicates that the double PEWMA chart is substantially less effective than the proposed chart by at least 34.4%. For example, the proposed scheme is more efficient than the double PEWMA by more than 47% in terms of PCI and ARARL when $\lambda = 0.2$ (cf. Table III).

TABLE IV
ARL COMPARISON OF POISSON EWMA CHARTS ($\mu_0 = 4,\ \lambda = 0.3$)

	Classical PEWMA	Double PEWMA	RR PEWMA	RR HS PEWMA	RR FIR PEWMA
μ	*L =2.735*	*L =2.462*	*L =2.213*	*L =2.207*	*L =2.210*
4.0	200.36	200.10	200.17	200.82	200.64
4.5	58.52	58.59	64.96	61.42	61.71
5.0	23.68	21.37	25.75	23.17	22.03
5.5	12.27	10.92	13.57	11.19	9.56
6.0	7.66	6.91	8.61	6.36	5.23
6.5	5.37	4.86	6.12	4.08	3.13
7.0	4.10	3.67	4.85	2.86	2.21
7.5	3.18	2.96	4.04	2.14	1.56
8.0	2.68	2.47	3.47	1.66	1.17
AQEL	31.428	28.624	36.866	24.604	20.006
PCI	1.571	1.431	1.843	1.230	1.000
ARARL	1.579	1.440	1.856	1.231	1.000

TABLE V
ARL COMPARISON OF POISSON EWMA CHARTS ($\mu_0 = 4,\ \lambda = 0.4$)

	Classical PEWMA	Double PEWMA	RR PEWMA	RR HS PEWMA	RR FIR PEWMA
μ	*L =2.797*	*L =2.580*	*L =2.150*	*L =2.152*	*L =2.115*
4.0	200.40	200.26	200.68	200.47	200.25
4.5	61.31	61.21	68.88	66.62	66.61
5.0	25.77	23.41	28.35	26.34	24.44
5.5	13.51	12.00	14.81	12.57	10.81
6.0	8.43	7.45	9.19	7.04	5.71
6.5	5.68	5.21	6.56	4.49	3.55
7.0	4.24	3.97	5.05	3.12	2.24
7.5	3.38	3.14	4.18	2.33	1.61
8.0	2.82	2.67	3.57	1.80	1.19
AQEL	33.586	30.846	39.050	27.166	21.643
PCI	1.552	1.425	1.804	1.255	1.000
ARARL	1.574	1.449	1.838	1.262	1.000

TABLE VI
ARL COMPARISON OF POISSON EWMA CHARTS ($\mu_0 = 4,\ \lambda = 0.5$)

	Classical PEWMA	Double PEWMA	RR PEWMA	RR HS PEWMA	RR FIR PEWMA
μ	*L =2.855*	*L =2.674*	*L =2.053*	*L =2.060*	*L =2.063*
4.0	200.22	200.34	200.10	200.46	200.08
4.5	65.36	62.23	72.00	71.17	70.10
5.0	28.67	24.71	30.80	29.31	26.97
5.5	15.16	12.75	15.90	14.15	12.10
6.0	9.08	7.87	9.87	7.86	6.21
6.5	6.23	5.40	6.82	5.04	3.84
7.0	4.51	3.99	5.26	3.40	2.50
7.5	3.47	3.22	4.29	2.48	1.72
8.0	2.89	2.70	3.63	1.90	1.26
AQEL	36.065	31.899	40.931	29.879	23.636
PCI	1.526	1.350	1.732	1.264	1.000
ARARL	1.548	1.375	1.767	1.271	1.000

C. Proposed vs. the simple $\frac{2}{2}$ PEWMA

We studied the application of a simple *two-out-of-two* runs rule (RR) on the classical PEWMA chart. For the classical

EWMA, the applications of runs rules have significantly improve the performance of the charts [6]. This however, is not the case with the PEWMA chart. The application of a simple $\frac{2}{2}$ rule on the later is worse than the classical PEWMA chart (cf. Tables I-VI). It is the least performing scheme in terms of ARL, AEQL, PCI and ARARL, in detecting mean changes in Poisson processes.

D. Proposed vs. the simple $\frac{2}{2}$ PEWMA with Head-Start

In this subsection, we analyzed the performance of the simple *two-out-of-two* runs rule on PEWMA with an initial head start [4], [1]. This model is very similar to the proposed scheme and is obtained by simultaneously using 50% head start on the two-sided PEWMA control statistic. The simulated results are displayed in column 6 of Tables I-VI. There is significant gain in the out-of-control ARL performance of this model dominating the classical, double and the simple $\frac{2}{2}$ runs rule PEWMA charts when $\delta \geq 1$, but not as efficient as the proposed scheme (cf. Table I-VI). From the overall point of view, the proposed scheme is more effective in detecting changes in Poisson processes than this model by more than 15% in terms of the AEQL, PCI and ARARL.

E. Decrease in Poisson Process Mean

All the above comparisons are for detection of increase in the process mean. However, changes in Poisson process could be due to decreases in mean. Using the above setting, we investigate the performance of the proposed chart relative to others and the results are displayed in form of ARL curves in Figs. 1 to 3.

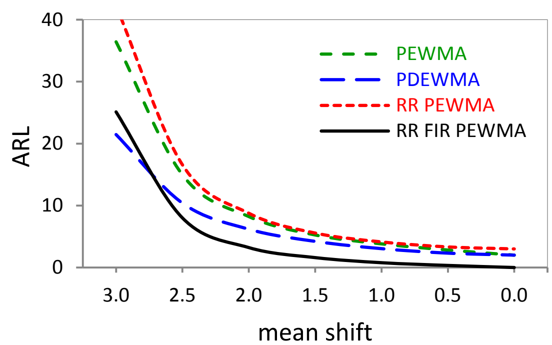

Fig. 1 ARL curves: decreases in process mean ($\mu_0 = 4,\ \lambda = 0.2$)

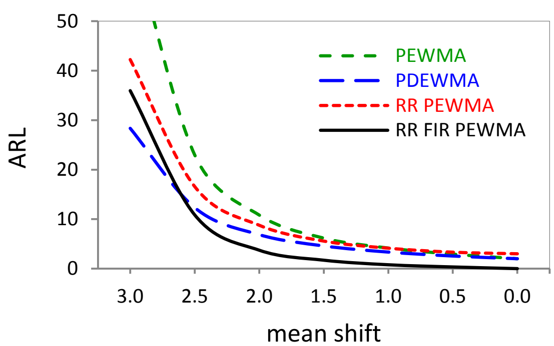

Fig. 2 ARL curves: decreases in process mean ($\mu_0 = 4,\ \lambda = 0.3$)

Fig. 1 shows that the simple $\frac{2}{2}$ runs rule PEWMA chart is the least performing chart in detecting decreases in the Poisson process mean shift. The proposed chart outperforms all other charts we investigated except in a very small region when the mean shift is less than one. In Figs. 2 and 3, the classical PEWMA is less effective than other charts. Overall, the proposed chart provides the best sensitivity to decreases in the Poisson mean processes.

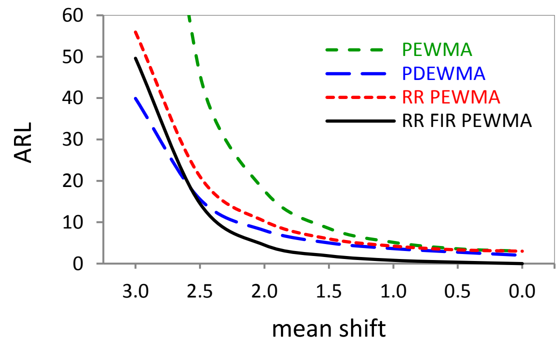

Fig. 3 ARL curves: decreases in process mean ($\mu_0 = 4,\ \lambda = 0.4$)

V. Conclusion and Recommendation

We have presented the joint use of two techniques, the simple two out of two runs rule and the FIR feature, for the Poisson EWMA chart to enhance the detection ability of the control chart. The addition of runs rule have greatly reduced the false alarm rates commonly associated with the FIR feature in EWMA control chart and reasonably maintaining low out-of-control ARL values. For simplicity, we have used an ease and one of the least performing runs rule.

Further studies on the use of more sensitive runs rules schemes such as the two out of three with FIR is recommended. The poor performance of the runs rule based PEWMA without the FIR feature may not be far from the discrete nature of the Poisson distribution.

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