Implementation of Modified SIRK Method on Solving Stiff Ordinary Differential Equations

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***Abstract*—**Stiff initial value problems with systems of ordinary differential equations arise from many applications in engineering and physics. This paper is focused on comprehensive derivation and implementation of a variant of the traditional Singly Implicit Runge Kutta Method for solving stiff differential equations system, which is to be called Modified Singly Implicit Runge Kutta Method (abbreviated to Modified SIRK). The method is studied for stage order up to order six. Some numerical experiments have been presented to show the potential of the approach in comparison with results by existence methods.

***Keywords*—**Singly Implicit Runge Kutta, stiff equations, initial value problems, Runge Kutta

I. Introduction

THIS paper is concerned with approximating solutions for system of first order differential equations in the form of equation (1).

$$\begin{aligned} \frac{du_1}{dt} &= f_1(t, u_1, u_2, \cdots, u_n) \\ \frac{du_2}{dt} &= f_2(t, u_1, u_2, \cdots, u_n) \\ &\vdots \\ \frac{du_n}{dt} &= f_n(t, u_1, u_2, \cdots, u_n) \end{aligned} \tag{1}$$

for $c \le t \le d$ subject to the initial conditions

$$u_1(c) = \alpha_1, u_2(c) = \alpha_2, \cdots, u_n(c) = \alpha_n. \tag{2}$$

The initial value problems (IVP) (1)-(2) are assumed to have a unique solution, under some definitions of IVP stated in [1].

In the literature of numerical methods for (1), some IVP are referred as stiff problems. Since 1952, numerical methods for stiff problems have been studied extensively. One of the characteristics of these problems is that they are extremely hard to solve by standard explicit step by step methods.

The stiff problems arise from many applications in mathematics and engineering such as electrical circuits and vibrations. In conjunction with the increasing on these applications, numbers of numerical methods have been introduced and upgraded to improve the accuracy of the approximation solutions. This paper is focused on comprehensive implementation of Modified Singly Implicit Runge Kutta Method (Modified SIRK) on solving stiff initial value problems.

Implicit Runge Kutta (IRK) formulation, which is known with ability to approximate with A-stable of stability condition for arbitrarily high-order was studied and faced problem of expensive cost for implementation. This problem then led to the search for special types of IRK methods. Part of the IRK types that have been proposed are Diagonally Implicit Runge Kutta Method by Alexander in 1977, and Singly Implicit Runge Kutta Method by Butcher and Burrage in 1976 [4],[8] and [10].

Butcher concluded that the most efficient way when using implementation of IRK are those whose characteristics polynomial of Runge Kutta matrix has a single real s-fold zero [2] and [4]. This method is then extended by Burrage in 1979 [8]. Burrage introduced two classes of methods which are Semi-Implicit Runge Kutta methods and Singly-Implicit methods. The Semi-Implicit Runge Kutta methods are methods whose Runge Kutta matrix is a lower triangular. However, disadvantage of this type of method is on difficulty to construct with large number of stage and high order. This difficulty existed as a result of requirement on modification of the order condition. The Singly-Implicit Runge Kutta (SIRK) method is a method with Runge Kutta matrix have non-lower triangular structure and a characteristic polynomial with single s-fold zero. Details on this Singly-Implicit RK method can be referred to [8].

Two years later, Burrage, Butcher and Chipman introduced an algorithm known as STRIDE, which was suggested for numerical computation of ordinary differential equations by Singly-Implicit Runge Kutta methods [7]. This algorithm was based on [8], which was corrected eight years later by Butcher [6]. In 1993, Moore and Flaherty introduced modification on implementation of SIRK which to be called as Modified SIRK in this paper [9]. Modified SIRK eliminated a linear system solution that is required in the traditional SIRK introduced by Butcher and Burrage [4] and [8]. The focus of study by Moore and Flaherty was on the implementation of Modified SIRK in time discretization of parabolic differential equations. There is no comprehensive details on derivation and implementation of Modified SIRK, as alternative in approximating stiff IVP. This paper is targeting to provide comprehensive study on derivation and implementation of Modified SIRK on solving stiff problems plus its advantages, and comparison to existed

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methods. The method is tested on different stage order of Modified SIRK, which is up to stage order of six.

II. Modified Singly-Implicit Runge Kutta Method (Modified SIRK)

A. SIRK Equation

IRK method at stage s can be represented by vectors $\hat{B}, \hat{C},$ and matrix A as stated in [3]. The matrix A is made to have characteristic polynomial $(z-\lambda)^s$, while $\hat{C}_1, \hat{C}_2, ..., \hat{C}_s$ are selected as $\lambda\xi_1, \lambda\xi_2, ..., \lambda\xi_s$. The entries of matrix A, which are $a_{i1}, a_{i2}, ..., a_{is}$ at $i = 1, 2, ..., s$, and entries of vector $\hat{B}$, which are $\hat{B}_1, \hat{B}_2, ..., \hat{B}_s$, chose to satisfy certain linear equations which can be referred from Butcher's study [3]. A bidiagonal matrix A with nonzero elements of λ as shown in (2), is a result of applying transformation $\overline{T}$ on matrix A.

$$\overline{T}^{-1} A \overline{T} = \begin{bmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ -\lambda & \lambda & & & \\ 0 & & & & \\ \vdots & \ddots & & & 0 \\ 0 & \cdots & 0 & -\lambda & \lambda \end{bmatrix} \quad (3)$$

The proof and details of this transformed IRK at s - stage can be referred to [3]. The variant of the traditional SIRK mainly based on the implementation of an equation in Modified SIRK, which defined the relationship of initial conditions and functions $f(t, u_1, u_2, \cdots, u_n)$ of the problem statement, together with transformation $\overline{T}$ and matrix A of IRK. This equation is to be called as SIRK equation in this paper. In general, by letting the approximation solution of type of problem stated in (1) as vector V with entries of $v_1, v_2, \cdots, v_n$, while matrix of A^{-1} based on matrix A, vector of initial conditions of problem statement as U_c, and functions $f(t, u_1, u_2, \cdots, u_n)$ of the problem statement as vector F, the SIRK equation of this problem can be represented as (4).

$$A^{-1}V - A^{-1}\underline{1}U_c = kF \ , \quad (4)$$

with k as the step size of the problem and $\underline{1}$ as unit vector of elements 1. It should be noted that the dimension of system of SIRK equation of equation (4) relies on value of s - stage order of Modified SIRK. Therefore, in matrix representation for s - stage order of Modified SIRK , the SIRK equation is written as equation (5).

The notations of $v_i^{(j)}$ and $f_i(c + \hat{c}_j, u_1, \cdots, u_n)^{(j)}$ at $i = 1, 2, ..., n$ and $j = 1, 2, ..., s$ corresponding to the values of variables at i of Modified SIRK method at stage order of s - stage. Instead of directly solving equation (5), Modified SIRK is solving the transformed of this equation.

$$\begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \ldots & a_{ss} \end{bmatrix}^{-1} \begin{bmatrix} v_1^{(1)} & \cdots & v_n^{(1)} \\ \vdots & & \vdots \\ v_1^{(s)} & \ldots & v_n^{(s)} \end{bmatrix} - \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \ldots & a_{ss} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^T$$

$$= kF, \quad (5)$$

$$F = \begin{bmatrix} f_1(c + \hat{C}_1 k, u_1, \cdots, u_n)^{(1)} & \cdots & f_n(c + \hat{C}_1, u_1, \cdots, u_n)^{(1)} \\ \vdots & & \vdots \\ f_1(c + \hat{C}_s, u_1, \cdots, u_n)^{(s)} & \cdots & f_n(c + \hat{C}_s, u_1, \cdots, u_n)^{(s)} \end{bmatrix}.$$

B. Transformation $\overline{T}$

The transformation $\overline{T}$ is based on Laguerre polynomials. The Laguerre polynomials can be represented recursively as (6).

$$L_0{}'(\xi) = 0$$
$$L_0(\xi) = 1 \quad (6)$$
$$\left.\begin{aligned} L_m{}'(\xi) &= L_{m-1}{}'(\xi) - L_{m-1}(\xi) \\ L_m(\xi) &= L_{m-1}(\xi) + \frac{\xi}{m} L_m{}'(\xi) \end{aligned}\right\} \quad m = 1, 2, 3, ...$$

It is well known that for $m = 1, 2, 3, \ldots$, the L_m has m distinct positive real zeros.

$$\overline{T} = L_{j-1}(\xi_i), \qquad i, j = 1, 2, ..., s \quad (7)$$

By using these characteristics of Laguerre polynomials, and taking $\xi_1, \xi_2, ..., \xi_s$ be the distinct zeros of L_m for s positive integer, $\overline{T}$ and $\overline{T}^{-1}$ matrices are denoted by (7) and (8) respectively.

$$\overline{T}^{-1} = \frac{\xi_j L_{i-1}(\xi_j)}{s^2 L_{s-1}(\xi_j)^2}, \qquad i, j = 1, 2, ..., s \quad (8)$$

C. Derivation of Modified SIRK Method

Considered that we deal with single step, at $[c, c+k]$ of our problem stated in (1) - (2). The approximation solutions by Modified SIRK are determined by (4). Using the transformation $\overline{T}$ mentioned above, we multiply (4) by $\overline{T}^{-1}$. The results of this transformation on SIRK equation can be referred to (9). Then, the new representation for SIRK equation mentioned in (9) after applying the transformation $\overline{T}$ is represented by (10).

D. Selection of λ in Modified SIRK

One of the critical conditions in applying either SIRK or Modified SIRK is the selection of λ. Burrage and Butcher implemented Wolfbrandt's idea on selection of λ values which based on the type of stability of the problems [2] and [11]. Wolfbrandt had studied about stability regions of rational approximations where the approximation function has a single pole. He concluded that for stability of $\mathrm{A}(\gamma)$-stable, methods with γ sufficiently close to $\frac{\pi}{2}$ could be expected to perform well with stiff problems. By relying on these results, λ values in Modified SIRK are chose to satisfied stability with $\gamma \approx \frac{\pi}{2}$. In general, λ is defined as (11) for these methods.

$$\lambda = \frac{1}{\xi}, \tag{11}$$

with ξ as a root of Laguerre polynomial at degree s. Since there will be s numbers of ξ for Laguerre polynomial of degree s, we denoted ξ^* for selected root which leading the method with the stability at $\gamma \approx \frac{\pi}{2}$. Table I shows value of λ for Modified SIRK at stages one to six, and selected root of Laguerre polynomial at degree s, ξ^* that satisfied the chosen stability condition.

$$\overline{T}^{-1} A^{-1} \overline{T}\,\overline{T}^{-1} V - (1U_c)\overline{T}^{-1} A^{-1} = k\overline{T}^{-1} F$$

$$\overline{T}^{-1} A^{-1} \overline{T} = \begin{bmatrix} \frac{\xi_1 L_0(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_0(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \\ \vdots & \cdots & \vdots \\ \frac{\xi_1 L_{s-1}(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_{s-1}(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{bmatrix}^{-1} \begin{bmatrix} L_0(\xi_1) & \cdots & L_{s-1}(\xi_1) \\ \vdots & \cdots & \vdots \\ L_0(\xi_s) & \cdots & L_{s-1}(\xi_s) \end{bmatrix} = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \lambda & \lambda & \cdots & \lambda \end{bmatrix},$$

$$(1U_c)\overline{T}^{-1} A^{-1} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \\ \vdots & & \vdots \\ \alpha_1 & \dots & \alpha_n \end{bmatrix} \begin{bmatrix} \frac{\xi_1 L_0(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_0(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \\ \vdots & \cdots & \vdots \\ \frac{\xi_1 L_{s-1}(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_{s-1}(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{bmatrix}^{-1} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \\ \vdots & & \vdots \\ \alpha_1 & \dots & \alpha_n \end{bmatrix} (\frac{1}{\lambda}), \tag{9}$$

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \lambda & \lambda & \cdots & \lambda \end{bmatrix} \begin{bmatrix} \frac{\xi_1 L_0(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_0(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \\ \vdots & \cdots & \vdots \\ \frac{\xi_1 L_{s-1}(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_{s-1}(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \end{bmatrix} \begin{bmatrix} v_1^{(1)} & \cdots & v_n^{(1)} \\ \vdots & & \vdots \\ v_1^{(s)} & \dots & v_n^{(s)} \end{bmatrix} - (\frac{1}{\lambda}) \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \\ \vdots & & \vdots \\ \alpha_1 & \dots & \alpha_n \end{bmatrix} \tag{10}$$

$$= k \begin{bmatrix} \frac{\xi_1 L_0(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_0(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \\ \vdots & \cdots & \vdots \\ \frac{\xi_1 L_{s-1}(\xi_1)}{s^2 L_{s-1}(\xi_1)^2} & \cdots & \frac{\xi_s L_{s-1}(\xi_s)}{s^2 L_{s-1}(\xi_s)^2} \end{bmatrix} \begin{bmatrix} f_1(c+\hat{C}_1 k, u_1, \cdots, u_n)^{(1)} & \cdots & f_n(c+\hat{C}_1, u_1, \cdots, u_n)^{(1)} \\ \vdots & & \vdots \\ f_1(c+\hat{C}_s, u_1, \cdots, u_n)^{(s)} & \cdots & f_n(c+\hat{C}_s, u_1, \cdots, u_n)^{(s)} \end{bmatrix}.$$

TABLE I

VALUES OF λ FOR s - STAGE MODIFIED SIRK

s	ξ^*	λ
1	1	1
2	3.4142	0.2929
3	2.2943	0.4359
4	1.7458	0.5728
5	3.5964	0.2781
6	2.9927	0.3341

III. NUMERICAL RESULTS

The method is tested on stiff problems, for subintervals of $N = 10$ to $N = 1000$. For each of problem, relative errors and order of accuracy are measured at each value of tested subinterval. Matlab software is applied as the programming language in analyzing the method on the tested stiff problems.

A. *Example 1*

$$x' = -1002x + 1000y^2,$$

$$y' = x - y(1+y),$$

where the initial conditions are $x(0)=1$ and $y(0)=1$, at $t\in[0,1]$. The exact solutions of problem are $x(t)=e^{-2t}$ and $y(t)=e^{-t}$.

TABLE II
ERROR VALUES OF EXAMPLE 1 BY s -STAGE MODIFIED SIRK FOR N=1000 AT SELECTED t VALUES

s	t	Exact		Approx		Rel. Errors	
		x	y	X	Y	E1	E2
1	0.4	4.4933E-01	6.7032E-01	4.4951E-01	6.7045E-01	1.8039E-04	1.3421E-04
	0.6	3.0119E-01	5.4881E-01	3.0138E-01	5.4898E-01	1.8122E-04	1.6480E-04
	0.8	2.0190E-01	4.4933E-01	2.0206E-01	4.4951E-01	1.6190E-04	1.7989E-04
2	0.4	4.4933E-01	6.7032E-01	4.4933E-01	6.7032E-01	1.4739E-08	1.0898E-08
	0.6	3.0119E-01	5.4881E-01	3.0119E-01	5.4881E-01	1.4770E-08	1.3378E-08
	0.8	2.0190E-01	4.4933E-01	2.0190E-01	4.4933E-01	1.3177E-08	1.4598E-08
3	0.4	4.4933E-01	6.7032E-01	4.4933E-01	6.7032E-01	1.1038E-11	8.1026E-12
	0.6	3.0119E-01	5.4881E-01	3.0119E-01	5.4881E-01	1.1029E-11	9.9413E-12
	0.8	2.0190E-01	4.4933E-01	2.0190E-01	4.4933E-01	9.8241E-12	1.0844E-11
4	0.4	4.4933E-01	6.7032E-01	4.4933E-01	6.7032E-01	2.8234E-12	2.1098E-12
	0.6	3.0119E-01	5.4881E-01	3.0119E-01	5.4881E-01	2.8396E-12	2.5903E-12
	0.8	2.0190E-01	4.4933E-01	2.0190E-01	4.4933E-01	2.5383E-12	2.8272E-12
5	0.4	4.4933E-01	6.7032E-01	4.4933E-01	6.7032E-01	7.7821E-12	5.8117E-12
	0.6	3.0119E-01	5.4881E-01	3.0119E-01	5.4881E-01	7.8283E-12	7.1377E-12
	0.8	2.0190E-01	4.4933E-01	2.0190E-01	4.4933E-01	6.9961E-12	7.7897E-12
6	0.4	4.4933E-01	6.7032E-01	4.4933E-01	6.7032E-01	2.7439E-12	2.0508E-12
	0.6	3.0119E-01	5.4881E-01	3.0119E-01	5.4881E-01	2.7635E-12	2.5212E-12
	0.8	2.0190E-01	4.4933E-01	2.0190E-01	4.4933E-01	2.4715E-12	2.7529E-12

TABLE III
RATE OF CONVERGENCES OF EXAMPLE 1 FOR SELECTED N VALUES

s	N	x		y	
		Error	β	Error	β
1	10	1.3340E-02		1.7685E-02	
	20	6.7259E-03	0.9879	9.0212E-03	0.9712
	40	3.3770E-03	0.9940	4.5569E-03	0.9853
	80	1.6920E-03	0.9970	2.2902E-03	0.9926
2	10	1.1204E-04		1.5078E-04	
	20	2.7847E-05	2.0084	3.7508E-05	2.0072
	40	6.9379E-06	2.0049	9.3545E-06	2.0034
	80	1.7303E-06	2.0035	2.3359E-06	2.0017
3	10	6.7809E-06		9.0735E-06	
	20	8.7095E-07	2.9608	1.1666E-06	2.9594
	40	1.1033E-07	2.9807	1.4796E-07	2.9790
	80	1.3873E-08	2.9915	1.8632E-08	2.9893
4	10	6.6173E-07		8.7774E-07	
	20	4.4515E-08	3.8939	5.9097E-08	3.8927
	40	2.8872E-09	3.9465	3.8381E-09	3.9446
	80	1.8382E-10	3.9733	2.4482E-10	3.9706
5	10	1.5666E-09		1.9110E-09	
	20	5.0197E-11	4.9638	6.1303E-11	4.9622
	40	1.3564E-12	5.2098	1.6325E-12	5.2308
	80	4.1669E-13	1.7027	5.7221E-13	1.5124
6	10	1.0386E-10		1.1981E-10	
	20	1.6842E-12	5.9464	1.9352E-12	5.9522
	40	5.3124E-14	4.9865	7.8326E-14	4.6269
	80	1.6179E-13	-1.6067	2.2082E-13	-1.4953

This example is taken from [10], which conducted study on numerical solutions for stiff problems. The details of relative errors produced by this Modified SIRK are presented in Table II while Table III represented values of rate of convergence, β. The values of β calculated at $t=1.0$, for respective s-stage of Modified SIRK. According to Table II, the approximations produced by Modified SIRK are comparable with the numerical results generated by proposed method mentioned in [10].

B. *Example 2*

$$u_1' = 32u_1 + 66u_2 + \frac{2}{3}t + \frac{2}{3},$$

$$u_2' = -66u_1 - 133u_2 - \frac{1}{3}t - \frac{1}{3},$$

where the initial conditions are $u_1(0)=\frac{1}{3}$ and $u_2(0)=\frac{1}{3}$, at $t\in[0,1]$. The exact solutions of problem are

$$u_1(t) = \frac{2}{3}t + \frac{2}{3}e^{-t} - \frac{1}{3}e^{-100t} \quad \text{and}$$

$$u_2(t) = -\frac{1}{3}t - \frac{1}{3}e^{-t} + \frac{2}{3}e^{-100t}.$$

This example is taken from [1] which emphasized on methods used to solve problem involving stiff differential equations. The details of relative errors produced by Modified SIRK are presented in Table IV. According to Table IV, the approximations produced by Modified SIRK are better than approximations produced by methods mentioned in [1].

TABLE IV

ERROR VALUES OF EXAMPLE 2 BY s - STAGE MODIFIED SIRK FOR N=1000 AT SELECTED t VALUES

s	t	Exact		Approx		Rel. Errors	
		u1	u2	U1	U2	E1	E2
1	0.4	7.1355E-01	-3.5677E-01	7.1364E-01	-3.5682E-01	8.9325E-05	4.4663E-05
	0.6	7.6587E-01	-3.8294E-01	7.6598E-01	-3.8299E-01	1.0971E-04	5.4853E-05
	0.8	8.3289E-01	-4.1644E-01	8.3301E-01	-4.1650E-01	1.1977E-04	5.9883E-05
2	0.4	7.1355E-01	-3.5677E-01	7.1355E-01	-3.5677E-01	7.2296E-09	3.6148E-09
	0.6	7.6587E-01	-3.8294E-01	7.6587E-01	-3.8294E-01	8.8786E-09	4.4393E-09
	0.8	8.3289E-01	-4.1644E-01	8.3289E-01	-4.1644E-01	9.6922E-09	4.8461E-09
3	0.4	7.1355E-01	-3.5677E-01	7.1355E-01	-3.5677E-01	5.5347E-12	2.7673E-12
	0.6	7.6587E-01	-3.8294E-01	7.6587E-01	-3.8294E-01	6.9629E-12	3.4819E-12
	0.8	8.3289E-01	-4.1644E-01	8.3289E-01	-4.1644E-01	7.8346E-12	3.9174E-12
4	0.4	7.1355E-01	-3.5677E-01	7.1355E-01	-3.5677E-01	1.7762E-12	8.8829E-13
	0.6	7.6587E-01	-3.8294E-01	7.6587E-01	-3.8294E-01	2.5056E-12	1.2528E-12
	0.8	8.3289E-01	-4.1644E-01	8.3289E-01	-4.1644E-01	3.1923E-12	1.5963E-12
5	0.4	7.1355E-01	-3.5677E-01	7.1355E-01	-3.5677E-01	5.3094E-12	2.6519E-12
	0.6	7.6587E-01	-3.8294E-01	7.6587E-01	-3.8294E-01	7.3649E-12	3.6803E-12
	0.8	8.3289E-01	-4.1644E-01	8.3289E-01	-4.1644E-01	9.2994E-12	4.6471E-12
6	0.4	7.1355E-01	-3.5677E-01	7.1355E-01	-3.5677E-01	1.7229E-12	8.6209E-13
	0.6	7.6587E-01	-3.8294E-01	7.6587E-01	-3.8294E-01	2.4392E-12	1.2194E-12
	0.8	8.3289E-01	-4.1644E-01	8.3289E-01	-4.1644E-01	3.1111E-12	1.5549E-12

Based on relative errors produced by tested stiff problems, the errors produced improve as the Modified SIRK considered for higher order of stage and as stepsize getting close to zero. Table III shown that the order of converge for Modified SIRK satisfied for each stage order. However, as the errors getting smaller for higher order of s -stage, the values of β varied. These situations are due to error values which are too small, together with high possibility of machine errors' involvement.

IV. CONCLUSION

Based on the numerical results, the relative errors produced reflected that the Modified SIRK should be considered as alternative method on solving stiff problems. The flexibility of achieving better approximations by increasing the stage order of Modified SIRK, without causing an increase on cost of computation, leading to one of advantages of this method. Besides, this method allows application at large number of system. However, the challenges of applying this method exist in formulation of Jacobian matrix. In this study, Newton's Method used to deal with nonlinearity of the systems. For the recommendation, Modified SIRK is a potential method that should be considered in dealing with stiff problems. Currently, this method is studying as one of the alternative methods for time discretization with combination of finite element methods for space discretization. It is expected that the approximations of further studies are improved due to the characteristics of Modified SIRK.

ACKNOWLEDGMENT

The author wishes to sincerely thank Assoc. Prof. Thanh Tran for the helps throughout the preparation of this paper.

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